A discrete Schrödinger spectral problem and associated evolution equations

M. Boiti, M. Bruschi†, F. Pempinelli and B. Prinari
Dipartimento di Fisica dell'Università and Sezione INFN,
Lecce, Italy
†Dipartimento di Fisica dell'Università di Roma "La Sapienza",
Roma, Italy

11 June, 2002

Abstract

A recently proposed discrete version of the Schrödinger spectral problem is considered. The whole hierarchy of differential-difference nonlinear evolution equations associated to this spectral problem is derived. It is shown that a discrete version of the KdV, sine-Gordon and Liouville equations are included and that the so called 'inverse' class in the hierarchy is local. The whole class of related Darboux and Bäcklund transformations is also exhibited.

1 Introduction

In recent years there has been a growing interest in the field of discrete dynamical systems, i.e. systems that can be described by ordinary and\or partial, generally nonlinear, difference equations.

Such systems arise and play an important role in a very large number of contexts and have an extensive range of applications: mathematical physics, chaos, fractals, disordered systems, biology, optics, economics, statistical physics, numerical analysis, discrete geometry, cellular automata, quantum field theory and so on[1].

Even if powerful analytic tools were developed in the last decades to deal with difference equations, due to the nonlinearity of the systems of interest, no technique of solution is available for most of such dynamical systems. Thus it is clear the relevance and the interest of any new integrable one.

In this paper we introduce a whole class of interesting nonlinear differential-difference evolution equations which correspond to isospectral deformations of the new Schrödinger discrete spectral problem introduced by [1] and investigated in [2] and which, therefore, are integrable via the inverse scattering method. We show that in this class are included the new discrete integrable version of the celebrated KdV found in [2], the discrete sine-Gordon and Liouville equations and a whole hierarchy of local equations, which as far as we know is new. We give also the explicit recurrence operators to construct the Darboux Transformations and the Bäcklund Transformations for the whole class. Of course the Bäcklund transformations are interesting 'per se' as difference-difference dynamical systems and moreover they can be suitable to construct special solitonic solutions for the considered differential-difference equations.

[1] The related literature is so large that it is impossible to quote even if in a very partial way. The interested reader can find a lot of relevant references in J. of Phys. A, **34** n. 48 (2001), Special issue: *Symmetries and integrability of difference equations (SIDE IV)*

2 The spectral problem

Let us consider the spectral problem

$$L(n)\psi(n;\lambda) = \lambda\psi(n;\lambda) \tag{2.1}$$

with

$$L(n) = E^2 + q(n)E^1 \tag{2.2}$$

where n is a discrete variable ($n \in \mathbb{Z}$), $\lambda \in \mathbb{C}$ is the spectral parameter and E^k is the 'shift' operator defined by

$$E^k\phi(n) = \phi(n+k), \qquad k = 0, \pm 1, \pm 2, ... \tag{2.3}$$

This spectral equation is the discretized version of the Schrödinger equation recently obtained by Shabat [1] iterating Darboux transformations of the continuous Schrödinger equation and its direct and inverse problems were first introduced and studied in [2].

In the following the 'potential' q and consequently the eigenfunction ψ will be considered depending as well on the continuous time variable t, while the spectral parameter is considered time independent, so that we can apply the inverse scattering method and consider the associated diffferential-difference nonlinear equations.

In the following we shall often use the shorthand notation

$$\phi(n) = \phi, \qquad \phi(n+k) = \phi_k, \qquad\quad k = \pm 1, \pm 2, ... \tag{2.4}$$

We will use also the first order difference operators

$$\Delta = E^1 - E^0 \tag{2.5}$$

$$\widetilde{\Delta} = E^1 + E^0 \tag{2.6}$$

and their inverses $\Delta^{-1}, \widetilde{\Delta}^{-1}$. Useful formulas for such difference operators and for difference equations are reported in Appendix A.

3 Isospectral hierarchy

We are looking for nonlinear discrete evolution equations associated with the isospectral deformations of the discrete Schrödinger operator L introduced in (2.2). They can be obtained from the Lax representation

$$\dot{L} = [L, M] \tag{3.1}$$

where $M(n, t, E^k)$ is an opportune shift operator and dot denotes time differentiation. We have

$$\dot{q}(n,t)E^1 = V(n,t)E^1 \tag{3.2}$$

where $V(n,t) = V(q, q_k)$ is a function depending on q and its shifted values.

To construct the hierarchy of isospectral nonlinear discrete evolution equations we can use a sort of dressing procedure. Precisely, we look for the *recursion operators* $\mathcal{M}$ and $\mathcal{L}$ which allow

to construct new admissible operators M' and V' in terms of known M and V. Following the technique introduced in [3] and [4], we start with the ansatz

$$M' = LM + AE^0 + BE^1 \tag{3.3}$$

where E^0 is the identity operator and A and B are functions, depending on the q and q_k, to be determined in such a way that

$$V'E^1 = [L, M'] \tag{3.4}$$

or explicitly

$$V'E^1 = q\,(A_1 - A)\,E^1 + (A_2 - A + qB_1 - q_1 B + qV_1)\,E^2 + (B_2 - B + V_2)\,E^3. \tag{3.5}$$

Now imposing for compatibility the vanishing of the terms in E^2 and E^3 and setting, for convenience,

$$C = A_1 - A \tag{3.6}$$

we get the two conditions

$$B_2 - B = -V_2 \tag{3.7}$$

$$C_1 + C = q_1 B - qB_1 - qV_1 \tag{3.8}$$

The general solution of the second order difference equation (3.7) reads

$$B = \overline{b} + \widehat{b}\,(-1)^n + \sum_{k=0}^{\infty} V_{2k+2} \tag{3.9}$$

where $\overline{b}, \widehat{b}$ are two arbitrary constants.

Taking into account (3.9) the general solution of the first order difference equation (3.8) reads

$$C = \widetilde{C} - q\sum_{k=0}^{\infty} V_{2k+1} - 2\sum_{k=1}^{\infty} (-1)^k\, q_k \sum_{j=0}^{\infty} V_{k+2j+1} \tag{3.10}$$

$$\widetilde{C} = \check{c}\,(-1)^n - \overline{b}\left(q + 2\sum_{k=1}^{\infty} (-1)^k\, q_k\right) + \widehat{b}\,(-1)^n\left(q + 2\sum_{k=1}^{\infty} q_k\right) \tag{3.11}$$

where $\check{c}$ is again an arbitrary constant.

Now, from (3.5), (3.6), (3.10) and (3.11) we get

$$V' = \widetilde{V} + \mathcal{L}V \tag{3.12}$$

where the *recursion operator* $\mathcal{L}$ is given by

$$\mathcal{L}V = -q\left(q\sum_{k=0}^{\infty} V_{2k+1} + 2\sum_{k=1}^{\infty} (-1)^k\, q_k \sum_{j=0}^{\infty} V_{k+2j+1}\right) \tag{3.13}$$

while

$$\widetilde{V} = q\left[\check{c}\,(-1)^n - \overline{b}\left(q + 2\sum_{k=1}^{\infty} (-1)^k\, q_k\right) + \widehat{b}\,(-1)^n\left(q + 2\sum_{k=1}^{\infty} q_k\right)\right]. \tag{3.14}$$

Note that starting from the trivial operators $M=0, V=0$ we get from (3.12) the non trivial starting point $\widetilde{V}$. Indeed, taking into account the fact that $\overline{b}, \widehat{b}, \check{c}$ are *arbitrary* constants, we have three different independent starting points

$$\check{V} = (-1)^n q \tag{3.15}$$

$$\overline{V} = q\left(q + 2\sum_{k=1}^{\infty} (-1)^k q_k\right) \tag{3.16}$$

$$\widehat{V} = (-1)^n q\left(q + 2\sum_{k=1}^{\infty} q_k\right) \tag{3.17}$$

so that the class of isospectral nonlinear discrete evolution equations associated with the spectral operator (2.1), (2.2) is a superposition of three hierarchies and reads

$$\dot{q} = \alpha(\mathcal{L})\check{V} + \beta(\mathcal{L})\overline{V} + \gamma(\mathcal{L})\widehat{V} \tag{3.18}$$

where α, β, γ are entire functions of their argument. The possibility of using negative powers of the recursion operator will be considered below.

It is easily seen that the recursion operator $\mathcal{L}$ in (3.13) can be written, by using the difference operators (2.5), (2.6) and their inverses (see Appendix A), in the more compact and elegant form

$$\mathcal{L} = -q\Delta\widetilde{\Delta}^{-1}qE^1\Delta^{-1}\widetilde{\Delta}^{-1}. \tag{3.19}$$

The explicit form of the recurrence operator $\mathcal{M}$ for M ($M' = \mathcal{M}M \iff V' = \mathcal{L}V$) can be easily obtained from the above formulas (see (3.3), (3.6), (3.9), (3.10), (3.11)).

Remark 1 *Note that all the M operators generated by the recursion (3.3) contain only positive powers of the shift operator E.*

Remark 2 *Due to the non-local character of the recursion operator $\mathcal{L}$ (see (3.13)) and taking into account the starting points (3.15), (3.16), (3.17), all the equations in the class (3.18) are apparently non local (except the rather trivial $\dot{q} = \check{V} = (-1)^n q$). However it is easily seen that $\dot{q} = \overline{V} = q\left(q + 2\sum_{k=1}^{\infty}(-1)^k q_k\right)$ and $\dot{q} = \widehat{V} = (-1)^n q\left(q + 2\sum_{k=1}^{\infty} q_k\right)$ imply respectively*

$$\dot{q}q_1 + q\dot{q}_1 = qq_1(q - q_1) \tag{3.20}$$

$$\dot{q}q_1 + q\dot{q}_1 = (-1)^n qq_1(q_1 + q) \tag{3.21}$$

which are local although not explicit. The first equation was derived and studied in ref. [2].

4 The 'inverse' class

As noted in the previous section, the equations in the class (3.18) correspond to positive shifts in the Lax operator M. In this section we investigate the class of isospectral nonlinear discrete evolution equations which corresponds to operators M constructed in terms of negative powers of the shift operator E.

These equations can be obtained using the inverse of the recursion operator $\mathcal{L}$, namely

$$\mathcal{L}^{-1} = -E^{-1}\Delta\widetilde{\Delta}\frac{1}{q}\Delta^{-1}\widetilde{\Delta}\frac{1}{q}. \tag{4.1}$$

Thus, given a valid couple M and V, one can obtain a new valid one $M' = \mathcal{M}^{-1}M$, $V' = \mathcal{L}^{-1}V$.

Starting from $V = 0$ and taking into account that an arbitrary constant c is in the *ker* of Δ we have

$$\widetilde{V} = \mathcal{L}^{-1}0 = -2\ c\left(\frac{1}{q_1} - \frac{1}{q_{-1}}\right) \tag{4.2}$$

and the whole 'inverse' class of isospectral nonlinear discrete evolution equations reads

$$\dot{q} = \eta\left(\mathcal{L}^{-1}\right)\widetilde{V} \tag{4.3}$$

where η is an arbitrary entire function of its argument.

However, in order to study the properties of this class of equations, it is more convenient to construct it explicitly through the ansatz

$$M^{(N)} = \sum_{k=1}^{N} A^{(k)}E^{k-N-1}. \tag{4.4}$$

Inserting the above expression in the Lax equation (3.1) it is easily seen that the functions $A^{(k)}$ $(k = 1, 2, ...N)$ are determined recursively by

$$qA_1^{(s+1)} - q_{s-N}A^{(s+1)} = A^{(s)} - A_2^{(s)}, \quad A^{(0)} = 0, \qquad s = 0, 1, ..., N-1. \tag{4.5}$$

The corresponding nonlinear discrete evolution equation reads

$$\dot{q} = A_2^{(N)} - A^{(N)}. \tag{4.6}$$

The general solution of (4.5) is, according to (A.14) and (A.17)

$$A^{(s+1)} = \frac{\overline{a}^{(s+1)} + \sum\limits_{k=0}^{\infty}\left(A_{k+2}^{(s)} - A_k^{(s)}\right)\dfrac{1}{q_k}\prod\limits_{j=\ 1}^{N-s} q_{-j+k+1}}{\prod\limits_{j=\ 1}^{N-s} q_{-j}} \tag{4.7}$$

where the constants $\overline{a}^{(s+1)}$ arise from the solution of the homogeneous part of the equation and can be chosen equal to zero except $\overline{a}^{(1)} = 1$ that gives a convenient starting point for the iteration.

In spite of the infinite series in (4.7) the $A^{(s+1)}$ are local for any N. In the Appendix B we solve explicitly the recursion relation proving by induction that

$$A^{(s+1)} = \frac{(-1)^s}{\prod\limits_{m=\ 1}^{N-s} q_{-m}} \sum_{k_1=-N}^{-1}\ \sum_{k_2=-N+1}^{k_1+1} \cdots \sum_{k_j=-N+j-1}^{k_{j-1}+1} \cdots \sum_{k_s=-N+s-1}^{k_{s-1}+1} Q_{k_1}\dots Q_{k_s}, \tag{4.8}$$

where

$$Q_k = \frac{1}{q_k q_{k+1}}. \tag{4.9}$$

For the reader's convenience, we write here explicitly the first three equations in the hierarchy.

1. For $N=1$ we get

$$\dot{q}=\left(\frac{1}{q_1}-\frac{1}{q_{-1}}\right) \tag{4.10}$$

which was implicitly considered in [2]. Note that setting

$$q=\frac{1}{u} \tag{4.11}$$

we get

$$\dot{u}=-u^2\left(u_1-u_{-1}\right). \tag{4.12}$$

2. For $N=2$ we get

$$\dot{q}=-\left(\frac{1}{q_1^2}\left(\frac{1}{q_2}+\frac{1}{q}\right)-\frac{1}{q_{-1}^2}\left(\frac{1}{q_{-2}}+\frac{1}{q}\right)\right) \tag{4.13}$$

which, using the position (4.11), becomes

$$\dot{u}=u^2\left(u_1^2\left(u_2+u\right)-u_{-1}^2\left(u_{-2}+u\right)\right) \tag{4.14}$$

3. For $N=3$ we get

$$\dot{q}=\left(\alpha_2-\alpha\right) \tag{4.15}$$

$$\alpha=\frac{1}{qq_{-1}^2}\left(\frac{1}{q_1q}+\frac{1}{qq_{-1}}+\frac{1}{q_{-1}q_{-2}}\right)+\frac{1}{q_{-1}^2q_{-2}}\left(\frac{1}{qq_{-1}}+\frac{1}{q_{-1}q_{-2}}+\frac{1}{q_{-2}q_{-3}}\right). \tag{4.16}$$

With the position (4.11) we get the polynomial equation

$$\dot{u}=-u^2\left(\beta_2-\beta\right) \tag{4.17}$$

$$\beta=uu_{-1}^2\left(u_1u+uu_{-1}+u_{-1}u_{-2}\right)+u_{-1}^2u_{-2}\left(uu_{-1}+u_{-1}u_{-2}+u_{-2}u_{-3}\right). \tag{4.18}$$

5 Discrete KdV, sine-Gordon and Liouville equations

Notice that the first equation (4.10) in the 'inverse' class of the equations related to the discrete Schrödinger operator (2.2) can be rewritten as

$$q_1\dot{q}+\dot{q}_1q=\frac{q}{q_2}-\frac{q_1}{q_{-1}} \tag{5.1}$$

and, then, by taking a linear combination with coefficients c and d of this equation with the equation (3.20) of the direct class we get

$$q_1\dot{q}+\dot{q}_1q=c\left(\frac{q}{q_2}-\frac{q_1}{q_{-1}}\right)+d(q_1-q)q_1q \tag{5.2}$$

which is the equation studied in [2] and that for $c=2d$ reduces to a discrete version of the KdV equation.

It was already shown in [5] that the discrete sine-Gordon and Liouville equations are included in the hierarchy of integrable equations related to the discrete Schrödinger spectral operator (2.2).

They can be recovered by using the special dressing method described above. In fact if, instead of starting in the iteration procedure from $V = 0$, $M = 0$, we start from

$$M = \partial_t, \quad V = -\dot{q} \tag{5.3}$$

we get

$$\dot{q} = V' = -\mathcal{L}\dot{q}, \tag{5.4}$$

which is a tricky equation since in order to extract from it an evolution equation it must be solved with respect to $\dot{q}$. In order to do this it is convenient to put

$$\dot{q} = -\gamma\Delta\widetilde{\Delta}e^{-2\varphi} \tag{5.5}$$

with γ an arbitrary constant and $\varphi = \varphi(n,t)$ a new function and

$$q = -e^{2\varphi_1}\widetilde{\Delta}P \tag{5.6}$$

with $P = P(n,t)$ to be determined. Inserting (5.5) and (5.6) into (5.4) we have

$$P_1^2 - P^2 = e^{-2(\varphi_1+\varphi_2)} - e^{-2(\varphi+\varphi_1)}. \tag{5.7}$$

With an opportune choice of the constant of integration and of the sign of P, we get

$$P = e^{-(\varphi+\varphi_1)}. \tag{5.8}$$

Therefore,

$$q = -e^{\varphi_1-\varphi} - e^{\varphi_1-\varphi_2} \tag{5.9}$$

and the differential-difference equation in φ can be obtained by imposing the compatibility between (5.5) and (5.9). We get

$$(\dot{\varphi}_1 - \dot{\varphi})e^{\varphi_1-\varphi} - (\dot{\varphi}_2 - \dot{\varphi}_1)e^{\varphi_1-\varphi_2} = \gamma e^{-2\varphi_2} - \gamma e^{-2\varphi}. \tag{5.10}$$

Inserting in it

$$\dot{\varphi}_1 - \dot{\varphi} = -\gamma e^{-(\varphi_1+\varphi)} + \Phi \tag{5.11}$$

with $\Phi = \Phi(n)$ to be determined we derive

$$\Phi_1 e^{-(\varphi_1+\varphi_2)} = \Phi e^{-(\varphi+\varphi_1)} \tag{5.12}$$

that can be integrated furnishing the evolution equation

$$\dot{\varphi}_1 - \dot{\varphi} = -\gamma e^{-(\varphi_1+\varphi)} + \gamma' e^{(\varphi_1+\varphi)} \tag{5.13}$$

which, up to a trivial change of function, for $\gamma' = \gamma$ is the discrete sine-Gordon and for $\gamma' = 0$ the Liouville equation.

Also the auxiliary spectral problem introduced in [5] for fixing the time evolution of φ can be recovered by using our dressing method. In fact we have

$$\dot{\psi} = -M'\psi \tag{5.14}$$

which, since

$$M' = LM + AE^0 + BE^1, \tag{5.15}$$

can be rewritten as

$$\dot{\psi} = -[L, M]\psi + \lambda M\psi - A\psi - B\psi_1. \tag{5.16}$$

Recalling that $M = \partial_t$ and denoting $\lambda = -1 - k^2$, we have

$$k^2\dot{\psi} = A\psi + (B - \dot{q})\psi_1. \tag{5.17}$$

Using the formulas obtained for A and B in section 3 and choosing equal to zero the constants of integration we get

$$A = -\widetilde{\Delta}^{-1}qE^1\Delta^{-1}\widetilde{\Delta}^{-1}V \tag{5.18}$$

and

$$B = -E^2\Delta^{-1}\widetilde{\Delta}^{-1}V. \tag{5.19}$$

Inserting the formulas for q and $\dot{q}$ obtained in (5.9) and (5.5) we recover the auxiliary spectral problem derived in [5]

$$k^2\dot{\psi} = \gamma e^{-\varphi_1-\varphi}\psi - \gamma e^{-2\varphi}\psi_1. \tag{5.20}$$

The second auxiliary problem introduced in [5] is obtained by closing the compatibility conditions with the discrete Schrödinger spectral problem (2.1).

6 Darboux and Bäcklund transformations

Let us consider besides (2.1) a second spectral problem with a different 'potential', namely let

$$\widetilde{L}\widetilde{\psi} = \lambda\widetilde{\psi} \tag{6.1}$$

with

$$\widetilde{L}(n, t) = E^2 + \widetilde{q}(n, t)E^1. \tag{6.2}$$

Let us introduce a Darboux transformation relating ψ and $\widetilde{\psi}$

$$\widetilde{\psi} = \mathcal{D}\psi \tag{6.3}$$

where $\mathcal{D}$ is an opportune shift operator depending on q and $\widetilde{q}$. This implies a relation between q and $\widetilde{q}$, called Bäcklund transformation, that can be expressed in the following operatorial form (see e.g. [3], [4])

$$\widetilde{L}\mathcal{D} - \mathcal{D}L = WE^1 = 0 \tag{6.4}$$

where the scalar operator W depends on q and $\widetilde{q}$ and their shifted values up to some order.

Now, following a technique introduced in [4], we look for the *recursion operators* Γ and Ω which allow to construct a valid couple of $\mathcal{D}'$ and W' from the supposed known $\mathcal{D}$, W, that is such that $\mathcal{D}' = \Gamma\mathcal{D}$, $W' = \Omega W$.

Consider the following *ansatz*

$$\mathcal{D}' = \widetilde{L}\mathcal{D} + FE^0 + GE^1 \tag{6.5}$$

where F and G are functions to be determined requiring that

$$W'E^1 = \widetilde{L}\mathcal{D}' - \mathcal{D}'L \tag{6.6}$$

for some W'. We have

$$W'E^1 = (\widetilde{q}F_1 - qF)\,E^1 + (F_2 - F + \widetilde{q}G_1 - q_1G + \widetilde{q}W_1)\,E^2 + (G_2 - G + W_2)\,E^3 \tag{6.7}$$

and imposing the vanishing of the terms in E^2 and E^3 we get the following conditions

$$G_2 - G = -W_2 \tag{6.8}$$
$$F_2 - F = q_1G - \widetilde{q}G_1 - \widetilde{q}W_1. \tag{6.9}$$

The general solution of the second order difference equation (6.8) reads

$$G = g + \overline{g}\,(-1)^n - E^2\Delta^{-1}\widetilde{\Delta}^{-1}W \tag{6.10}$$

where g, $\overline{g}$ are two arbitrary constants.

The general solution of the second order difference equation (6.9) reads

$$F = f + \overline{f}\,(-1)^n + \Delta^{-1}\widetilde{\Delta}^{-1}\,(q_1G - \widetilde{q}G_1 - \widetilde{q}W_1) \tag{6.11}$$

where f, $\overline{f}$ are two arbitrary constants.

Therefore, inserting (6.10) we have

$$F = \widetilde{F} + \overline{F} \tag{6.12}$$

where

$$\widetilde{F} = f + \overline{f}\,(-1)^n - gE^1\Delta^{-1}\widetilde{\Delta}^{-1}(\widetilde{q}_{-1} - q) - \overline{g}(-1)^nE^1\Delta^{-1}\widetilde{\Delta}^{-1}\,(\widetilde{q}_{-1} + q) \tag{6.13}$$

and

$$\overline{F} = E^1\Delta^{-1}\widetilde{\Delta}^{-1}\left\{-\widetilde{q}_{-1} + (\widetilde{q}_{-1}E^1 - q)E^1\Delta^{-1}\widetilde{\Delta}^{-1}\right\}W. \tag{6.14}$$

From (6.7), (6.12), (6.13) and (6.14) we get

$$W' = \widetilde{W} + \Lambda W \tag{6.15}$$

where

$$\widetilde{W} = \left(\widetilde{q}\widetilde{F}_1 - q\widetilde{F}\right) \tag{6.16}$$
$$\Lambda = \left(\widetilde{q}E^1 - q\right)E^1\Delta^{-1}\widetilde{\Delta}^{-1}\left\{-\widetilde{q}_{-1} + (\widetilde{q}_{-1}E^1 - q)E^1\Delta^{-1}\widetilde{\Delta}^{-1}\right\}. \tag{6.17}$$

Starting from the trivial operators $\mathcal{D} = 0, W = 0$ we get from (6.15) a non trivial Bäcklund transformation $\widetilde{W}$. Indeed, taking into account the arbitrariness of the constants in (6.13) we have four different independent starting points so that the class of Bäcklund transformations reads

$$\sum_{k=1}^{4}\alpha_k\,(\Lambda)\,W^{(k)} = 0 \tag{6.18}$$

where the $\alpha_k(\Lambda)$ are *arbitrary* entire functions of their argument and the 'elementary' Bäcklund transformations $W^{(k)}$ are given by

$$W^{(1)} = (\widetilde{q} - q) \tag{6.19}$$

$$W^{(2)} = (-1)^n (\widetilde{q} + q) \tag{6.20}$$

$$W^{(3)} = \left(\tilde{q}E^1 - q\right) E^1 \Delta^{-1} \widetilde{\Delta}^{-1} (\widetilde{q}_{-1} - q) \tag{6.21}$$

$$W^{(4)} = (-1)^n \left(\widetilde{q}E^1 + q\right) E^1 \Delta^{-1} \widetilde{\Delta}^{-1} (\widetilde{q}_{-1} + q). \tag{6.22}$$

The explicit form of the recurrence operator Γ for $\mathcal{D}$ can be easily obtained from the above formulas.

Remark 3 *In the limit $\widetilde{q} \to q$, as expected, the operator Λ becomes the operator $\mathcal{L}$.*

In fact let us note that Λ can be rewritten as

$$\Lambda = \left(q - \widetilde{q}E^1\right) \left\{ \sum_{k=0}^{\infty} E^{2k} (q_1 - \widetilde{q}E^1) \sum_{j=1}^{\infty} E^{2j} - \sum_{k=0}^{\infty} E^{2k} \widetilde{q} E^1 \right\}.$$

For $\widetilde{q} = q$ we get, recalling (A.10)

$$\Lambda_{\widetilde{q}=q} = \left(q - qE^1\right) \sum_{k=0}^{\infty} (-1)^k E^k q E^1 \Delta^{-1} \widetilde{\Delta}^{-1}$$

and, thanks to (A.8) we have

$$\Lambda_{\widetilde{q}=q} = \mathcal{L}.$$

7 Concluding remarks

To end this paper we want to outline a number of possible extensions and generalizations.

First of all we conjecture that the whole hierarchy here introduced is endowed with a double Hamiltonian structure. Finding such structure should allow us to exhibit an infinite number of commuting conservation laws for the whole hierarchy.

Moreover, the technique we used to derive our results (recurrence operators, Lax pairs and Darboux transformations) can be easily extended to recover differential-difference nonlinear evolution equations related with non isospectral deformations of the spectral problem, getting typically equations with n-dependent coefficient (see [7]).

The Bäcklund transformations and the discrete-discrete evolution equations that can be associated to this new spectral problem (see [2, 5]) deserve further investigation as integrable nonlinear iterated maps and moreover offer a good starting point for the introduction of new 'integrable' cellular automata (see e.g. [9]).

Finally, all these results could be generalized to the non-abelian case considering a matrix discrete Schrödinger operator and matrix differential-difference evolution equations (see e.g. [8]).

A Appendix

For the convenience of readers not so familiar with difference equations and operators we give here some useful formulas.

Let us define the first order difference operators

$$\Delta = E^1 - E^0 \tag{A.1}$$

$$\widetilde{\Delta} = E^1 + E^0 \tag{A.2}$$

where E^0 is the identity operator. The general solutions of the difference equations

$$\begin{aligned} \Delta X &= F, \qquad X = X(n),\ F = F(n) && \text{(A.3)} \\ \widetilde{\Delta} Y &= G \qquad Y = Y(n),\ G = G(n) && \text{(A.4)} \end{aligned}$$

can be written as

$$\begin{aligned} X &= c + \Delta^{-1} F && \text{(A.5)} \\ Y &= c(-1)^n + \widetilde{\Delta}^{-1} G && \text{(A.6)} \end{aligned}$$

where c is an arbitrary constant, i.e. not depending on n, and the inverses of Δ and $\widetilde{\Delta}$ are chosen as follows

$$\Delta^{-1} = -\sum_{k=0}^{\infty} E^k \tag{A.7}$$

$$\widetilde{\Delta}^{-1} = \sum_{k=0}^{\infty} (-1)^k E^k. \tag{A.8}$$

We can also construct higher order operators using these fundamental ones as the second order difference operator

$$\Delta\widetilde{\Delta} = E^2 - E^0 \tag{A.9}$$

and its inverse

$$\Delta^{-1}\widetilde{\Delta}^{-1} = -\sum_{k=0}^{\infty} E^{2k}. \tag{A.10}$$

Thus the general solution of the second order difference equation

$$\Delta\widetilde{\Delta} X = F \tag{A.11}$$

can be written as follows

$$X = c + \overline{c}(-1)^n + \Delta^{-1}\widetilde{\Delta}^{-1} F \tag{A.12}$$

where c and $\overline{c}$ are arbitrary constants.

Finally, let us consider the general homogeneous first order difference equation

$$A X_1 - B X = 0, \qquad X = X(n),\ A = A(n),\ B = B(n). \tag{A.13}$$

Its general solution can be written as follows

$$X = c \prod_{k=0}^{+\infty} \frac{A_k}{B_k}$$

where c is again an arbitrary constant.

Then for the non homogenous first order difference equation

$$AX_1 - BX = F, \qquad F = F(n) \tag{A.14}$$

we search a solution of the form

$$X = \prod_{k=0}^{+\infty} \frac{A_k}{B_k} G \tag{A.15}$$

with $G = G(n)$ to be determined. We get

$$G_1 - G = \frac{1}{B} \prod_{k=0}^{+\infty} \frac{B_k}{A_k} F \tag{A.16}$$

and therefore

$$X = c \prod_{k=0}^{+\infty} \frac{A_k}{B_k} - \sum_{j=0}^{+\infty} \frac{1}{A_j} \prod_{k=0}^{j} \frac{A_k}{B_k} F_j. \tag{A.17}$$

B Appendix

We choose $\overline{a}^{(1)} = 1$ and all other $\overline{a}^{(s+1)} = 0$. It is convenient to rewrite the recursion relation for the $A^{(s+1)}$ in terms of

$$B^{(s)} = A^{(s)} \prod_{j=1}^{N-s+1} q_{-j}. \tag{B.1}$$

We have

$$B_1^{(s+1)} - B^{(s+1)} = -\frac{B_2^{(s)}}{qq_1} + \frac{B^{(s)}}{q_{-N+s-1}q_{-N+s}}, \qquad B^{(0)} = 0 \tag{B.2}$$

and then, if we choose all integration constants zero,

$$B^{(s+1)} = \delta_{s,0} + \sum_{k=0}^{\infty} \left(\frac{B_{k+2}^{(s)}}{q_k q_{k+1}} - \frac{B_k^{(s)}}{q_{k-N+s-1}q_{k-N+s}} \right). \tag{B.3}$$

Now we prove by induction on s that for $s \geq 0$

$$B^{(s+1)} = (-1)^s \sum_{k_1=-N}^{-1} \sum_{k_2=-N+1}^{k_1+1} \cdots \sum_{k_j=-N+j-1}^{k_{j-1}+1} \cdots \sum_{k_s=-N+s-1}^{k_{s-1}+1} Q_{k_1} \dots Q_{k_s} \tag{B.4}$$

where

$$Q_k = \frac{1}{q_k q_{k+1}}. \tag{B.5}$$

From (B.3) we have

$$B^{(s+1)} = (-1)^{s+1}\beta_1 + (-1)^s \beta_2 \tag{B.6}$$

where

$$\beta_1 \;=\; \sum_{k_1=0}^{\infty} Q_{k_1} \sum_{k_2=-N}^{-1} \sum_{k_3=-N+1}^{k_2+1} \cdots \sum_{k_j=-N+j-2}^{k_{j-1}+1} \cdots \sum_{k_s=-N+s-2}^{k_{s-1}+1} Q_{k_2+k_1+2} \ldots Q_{k_s+k_1+2} \tag{B.7}$$

$$\beta_2 \;=\; \sum_{k_1=0}^{\infty} Q_{k_1-N+s-1} \sum_{k_2=-N}^{-1} \sum_{k_3=-N+1}^{k_2+1} \cdots \sum_{k_j=-N+j-2}^{k_{j-1}+1} \cdots \sum_{k_s=-N+s-2}^{k_{s-1}+1} Q_{k_2+k_1} \ldots Q_{k_s+k_1}. \tag{B.8}$$

In (B.7) we introduce $k'_j = k_1 + k_j + 2$ for $j = 2, \ldots, s$ and renaming $k'_j \to k_j$ we obtain

$$\beta_1 = \sum_{k_1=0}^{\infty} \sum_{k_2=-N+k_1+2}^{k_1+1} \cdots \sum_{k_j=k_1-N+j}^{k_{j-1}+1} \cdots \sum_{k_s=k_1-N+s}^{k_{s-1}+1} Q_{k_1} Q_{k_2} \ldots Q_{k_s}. \tag{B.9}$$

In (B.8) we introduce $k'_1 = k_1 - N + s - 1$ and $k'_j = k_1 + k_j$ for $j = 2, \ldots, s$ and then rename $k'_j \to k_j$

$$\beta_2 = \sum_{k_1=-N+s-1}^{\infty} \sum_{k_2=k_1-s+1}^{k_1-s+N} \sum_{k_3=k_1-s+2}^{k_2+1} \cdots \sum_{k_j=k_1+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_1-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s}.$$

Exchanging the first two sums we obtain

$$\beta_2 = \sum_{k_2=-N}^{\infty} \sum_{k_1=\max\{k_2+s-N,-N+s-1\}}^{k_2+s-1} \sum_{k_3=k_1-s+2}^{k_2+1} \cdots \sum_{k_j=k_1+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_1-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s}$$

and with $k_1 \leftrightarrow k_2$

$$\begin{aligned}
\beta_2 \;&=\; \sum_{k_1=-N}^{\infty} \sum_{k_2=\max\{k_1+s-N,-N+s-1\}}^{k_1+s-1} \sum_{k_3=k_2-s+2}^{k_1+1} \cdots \sum_{k_j=k_2+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_2-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s} \\
&=\; \sum_{k_1=-N}^{-1} \sum_{k_2=-N+s-1}^{k_1+s-1} \sum_{k_3=k_2-s+2}^{k_1+1} \cdots \sum_{k_j=k_2+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_2-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s} + \\
&\quad + \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+s-N}^{k_1+s-1} \sum_{k_3=k_2-s+2}^{k_1+1} \cdots \sum_{k_j=k_2+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_2-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s}.
\end{aligned} \tag{B.10}$$

Let us consider first of all the second term and exchange the sums from the left to the right. After $j-1$ inversions we have

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_{j-1}=k_1-N+(j-1)}^{k_{j-2}+1} \sum_{k_j=k_1+s-N}^{k_{j-1}+s-(j-1)} \sum_{k_{j+1}=k_j-s+j}^{k_{j-1}+1} \sum_{k_{j+2}=k_j+(j+2)-s-1}^{k_{j+1}+1} \cdots \sum_{k_s=k_j-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s}$$

and exchanging $\sum_{k_j}$ and $\sum_{k_{j+1}}$ one gets

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_{j-1}=k_1-N+(j-1)}^{k_{j-2}+1} \sum_{k_{j+1}=k_1-N+j}^{k_{j-1}+1} \sum_{k_j=k_1+s-N}^{k_{j+1}+s-j} \sum_{k_{j+2}=k_j+(j+2)-s-1}^{k_{j+1}+1} \cdots \sum_{k_s=k_j-1}^{k_{s-1}+1} Q_{k_1} \ldots Q_{k_s}$$

so that with $k_j \leftrightarrow k_{j+1}$

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_{j-1}=k_1-N+(j-1)}^{k_{j-2}+1} \sum_{k_j=k_1-N+j}^{k_{j-1}+1} \sum_{k_{j+1}=k_1+s-N}^{k_j+s-j} \sum_{k_{j+2}=k_{j+1}+(j+2)-s-1}^{k_j+1} \cdots \sum_{k_s=k_{j+1}-1}^{k_{s-1}+1} Q_{k_1} \dots Q_{k_s}$$

and we only have to observe that for $s = j+1$ the process ends giving for the last sum $\sum_{k_s=k_1+s-N}^{k_{s-1}+1}$.

From (B.6), (B.9) and (B.10) it then follows that

$$B^{(s+1)} = (-1)^s \sum_{k_1=-N}^{-1} \sum_{k_2=-N+s-1}^{k_1+s-1} \sum_{k_3=k_2-s+2}^{k_1+1} \cdots \sum_{k_j=k_2+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_2-1}^{k_{s-1}+1} Q_{k_1} \dots Q_{k_s}.$$

Again we exchange $\sum_{k_2}$ and $\sum_{k_3}$ getting

$$B^{(s+1)} = (-1)^s \sum_{k_1=-N}^{-1} \sum_{k_3=-N+1}^{k_1+1} \sum_{k_2=-N+s-1}^{k_3+s-2} \sum_{k_4=k_2-s+3}^{k_3+1} \cdots \sum_{k_j=k_2+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_2-1}^{k_{s-1}+1} Q_{k_1} \dots Q_{k_s}$$

and for $k_2 \leftrightarrow k_3$

$$B^{(s+1)} = (-1)^s \sum_{k_1=-N}^{-1} \sum_{k_2=-N+1}^{k_1+1} \sum_{k_3=-N+s-1}^{k_2+s-2} \sum_{k_4=k_3-s+3}^{k_2+1} \cdots \sum_{k_j=k_3+j-s-1}^{k_{j-1}+1} \cdots \sum_{k_s=k_3-1}^{k_{s-1}+1} Q_{k_1} \dots Q_{k_s}$$

so that after performing all the inversions we recover (B.4).

Acknowledgments

This work was partially supported by PRIN 2000 "Sintesi" and was performed in the framework of the INTAS project 99-1782.

References

[1] A. Shabat, in *Nonlinearity, Integrability and All That. Twenty Years After NEEDS '79*, ed. M. Boiti, L. Martina, F. Pempinelli, B. Prinari and G. Soliani (Singapore: World Scientific, 2000) 331

[2] M. Boiti, F. Pempinelli, B. Prinari and A. Spire,"An integrable discretization of KdV at large times" *Inverse Problems* **17** (2001) 515

[3] M. Bruschi and O. Ragnisco, "Existence of a Lax Pair for any member of the class of nonlinear evolution equations associated to the matrix Schroedinger spectral problem" *Lettere al Nuovo Cimento* **29** (1980) 321

[4] M. Bruschi and O. Ragnisco, "Nonlinear differential-difference equations, Backlund transformations and Lax technique" *Journal of Physics A: Math. Gen.* **14** (1981) 1075

[5] M.Boiti, F. Pempinelli, P. Prinari and A. Spire, "Integrable discretizations of the sine-Gordon equation" nlin.SI/0102012

[6] M. Bruschi and O. Ragnisco "Backlund transformations and Lax technique" *Lettere al Nuovo Cimento* **29** (1980) 331

[7] M. Bruschi and O. Ragnisco, “Nonlinear differential difference matrix equations with n-dependent coefficients” *Lettere al Nuovo Cimento* **31** (1981) 492

[8] M. Bruschi, D. Levi and O. Ragnisco, "Evolution equations associated to the discrete analogue of the matrix Schroedinger problem solvable by the Inverse Spectral Transform" *Journal of Math. Phys.* **21** (1981) 2463

[9] M. Bruschi, O. Ragnisco and P.M. Santini , “Integrable cellular automata” *Physics Letters A* **169** (1992) 151